

Hyper–digital logic

Summarized overview of its syntax and semantics

www.bucephalus.org

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Figure 1: Bit values and their algebra

Bit values

$\mathbb{B} := \{0, 1\}$ is the bit value class, where **0** is the zero bit and **1** the unit bit.

Bit value algebra

$\mathfrak{B} := \langle \mathbb{B}, \leq, \mathbf{0}, \mathbf{1}, \wedge, \vee, \bigwedge, \bigvee, - \rangle$ is the bit value algebra, where

$\beta_1 \leq \beta_2$ iff $\beta_1 = \mathbf{0}$ or $\beta_2 = \mathbf{1}$

$\beta_1 \wedge \beta_2 := \bigwedge \{\beta_1, \beta_2\}$

$\bigwedge \mathcal{B} := \begin{cases} \mathbf{0} & \text{if } \mathbf{0} \in \mathcal{B} \\ \mathbf{1} & \text{else} \end{cases}$

$-\beta := \begin{cases} \mathbf{0} & \text{if } \beta = \mathbf{1} \\ \mathbf{1} & \text{else} \end{cases}$

$\beta_1 \vee \beta_2 := \bigvee \{\beta_1, \beta_2\}$

$\bigvee \mathcal{B} := \begin{cases} \mathbf{1} & \text{if } \mathbf{1} \in \mathcal{B} \\ \mathbf{0} & \text{else} \end{cases}$

for all $\beta, \beta_1, \beta_2 \in \mathbb{B}$ and $\mathcal{B} \subseteq \mathbb{B}$.

We also write, for all $\beta_1, \dots, \beta_n \in \mathbb{B}$ with $n \geq 0$,

$\bigwedge_{i=1}^n \beta_i$ for $\bigwedge \{\beta_1, \dots, \beta_n\}$

and

$\bigvee_{i=1}^n \beta_i$ for $\bigvee \{\beta_1, \dots, \beta_n\}$

Theorem

\mathfrak{B} is a complete boolean algebra.

Figure 2: Bit tables and their algebras

Bit tables

For every set A and each natural number k we define

$$\mathbb{B}_A^k := \begin{cases} A & \text{if } k = 0 \\ \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} & \text{if } k > 0 \end{cases}$$

the **bit table** set of **carrier** A and **degree** k .

In our default notation for functions^a, each bit table $\Omega \in \mathbb{B}_A^k$ with $k \geq 1$ is then given by

$$\Omega = \left[\begin{array}{c} \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} \\ \omega \mapsto \Omega(\omega) \end{array} \right]$$

Similar to geometry, bit tables of small degree $k = 0, 1, 2, 3$ are also called *bit points*, *bit lines*, *bit squares* and *bit cubes*, respectively. In traditional propositional logic, bit squares are also known as *truth tables*.

Bit table diagrams

If both A and k are finite, we can represent each $\Omega \in \mathbb{B}_A^k$ by its **bit table diagram**. For example

(1) If $A = \{a, b\}$ and $k = 1$ then

(2) If $A = \{a, b\}$ and $k = 2$ then

(3) If $A = \{a\}$ and $k = 3$ then

$$\begin{array}{ccc} \begin{array}{|c|c|c|} \hline a & b & c \\ \hline \beta_1 & \beta_2 & \beta_3 \\ \hline \end{array} & := & \left[\begin{array}{c} A \longrightarrow \mathbb{B} \\ a \mapsto \beta_1 \\ b \mapsto \beta_2 \\ c \mapsto \beta_3 \end{array} \right] \\ \\ \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & \beta_3 \\ 1 & 1 & \beta_4 \\ \hline \end{array} & := & \left[\begin{array}{c} \mathbb{B}_A^1 \longrightarrow \mathbb{B} \\ \begin{array}{|c|c|} \hline a & b \\ \hline 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ \hline \end{array} \mapsto \beta_1 \\ \begin{array}{|c|c|} \hline a & b \\ \hline 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ \hline \end{array} \mapsto \beta_2 \\ \begin{array}{|c|c|} \hline a & b \\ \hline 0 & 1 \\ 1 & 1 \\ \hline \end{array} \mapsto \beta_3 \\ \begin{array}{|c|c|} \hline a & b \\ \hline 1 & 1 \\ \hline \end{array} \mapsto \beta_4 \end{array} \right] \\ \\ \begin{array}{|c|c|c|c|c|} \hline a & & & & \\ \hline 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline \beta_1 & \beta_2 & \beta_3 & \beta_4 & \\ \hline \end{array} & := & \left[\begin{array}{c} \mathbb{B}_A^2 \longrightarrow \mathbb{B} \\ \begin{array}{|c|c|} \hline a & \\ \hline 0 & 0 \\ 1 & 0 \\ \hline \end{array} \mapsto \beta_1 \\ \begin{array}{|c|c|} \hline a & \\ \hline 0 & 1 \\ 1 & 1 \\ \hline \end{array} \mapsto \beta_2 \\ \begin{array}{|c|c|} \hline a & \\ \hline 0 & 0 \\ 1 & 1 \\ \hline \end{array} \mapsto \beta_3 \\ \begin{array}{|c|c|} \hline a & \\ \hline 0 & 1 \\ 1 & 1 \\ \hline \end{array} \mapsto \beta_4 \end{array} \right] \end{array}$$

Bit table algebras

$\mathfrak{B}_A^k := \langle \mathbb{B}_A^k, \sqsubseteq_A^k, \perp_A^k, \top_A^k, \sqcap_A^k, \sqcup_A^k, \prod_A^k, \coprod_A^k, \neg_A^k \rangle$ is the **bit table algebra**, for each set A and $k \geq 1$, where

$$\Omega \sqsubseteq_A^k \Omega' \text{ iff } \Omega(\omega) \leq \Omega'(\omega) \text{ for all } \omega \in \mathbb{B}_A^{k-1}$$

$$\Omega \sqcap_A^k \Omega' := \left[\begin{array}{c} \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} \\ \omega \mapsto \Omega(\omega) \wedge \Omega'(\omega) \end{array} \right] \quad \Omega \sqcup_A^k \Omega' := \left[\begin{array}{c} \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} \\ \omega \mapsto \Omega(\omega) \vee \Omega'(\omega) \end{array} \right] \quad \neg_A^k \Omega := \left[\begin{array}{c} \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} \\ \omega \mapsto \neg \Omega(\omega) \end{array} \right]$$

$$\perp_A^k := \left[\begin{array}{c} \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} \\ \omega \mapsto \mathbf{0} \end{array} \right] \quad \top_A^k := \left[\begin{array}{c} \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} \\ \omega \mapsto \mathbf{1} \end{array} \right] \quad \prod_A^k \Gamma := \left[\begin{array}{c} \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} \\ \omega \mapsto \bigwedge \{ \Omega(\omega) \mid \Omega \in \Gamma \} \end{array} \right] \quad \coprod_A^k \Gamma := \left[\begin{array}{c} \mathbb{B}_A^{k-1} \longrightarrow \mathbb{B} \\ \omega \mapsto \bigvee \{ \Omega(\omega) \mid \Omega \in \Gamma \} \end{array} \right]$$

for all $\Omega, \Omega' \in \mathbb{B}_A^k$ and $\Gamma \subseteq \mathbb{B}_A^k$.

Using bit table diagrams and taking $A = \{a, b\}$ and $k = 2$ for example, the operations are

$$\begin{array}{ccc} \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & \beta_3 \\ 1 & 1 & \beta_4 \\ \hline \end{array} \sqsubseteq_A^2 \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & \gamma_1 \\ 1 & 0 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 1 & 1 & \gamma_4 \\ \hline \end{array} \text{ iff } \left(\begin{array}{l} \beta_1 \leq \delta_1 \text{ and} \\ \beta_2 \leq \delta_2 \text{ and} \\ \beta_3 \leq \delta_3 \text{ and} \\ \beta_4 \leq \delta_4 \end{array} \right) & \perp_A^2 = \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ \hline \end{array} & \top_A^2 = \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ \hline \end{array} \\ \\ \neg_A^2 \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & \beta_3 \\ 1 & 1 & \beta_4 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & -\beta_1 \\ 1 & 0 & -\beta_2 \\ 0 & 1 & -\beta_3 \\ 1 & 1 & -\beta_4 \\ \hline \end{array} \quad \prod_A^2 \Gamma = \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \wedge \gamma_1 \\ 1 & 0 & \beta_2 \wedge \gamma_2 \\ 0 & 1 & \beta_3 \wedge \gamma_3 \\ 1 & 1 & \beta_4 \wedge \gamma_4 \\ \hline \end{array} \quad \coprod_A^2 \Gamma = \begin{array}{|c|c|c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \vee \gamma_1 \\ 1 & 0 & \beta_2 \vee \gamma_2 \\ 0 & 1 & \beta_3 \vee \gamma_3 \\ 1 & 1 & \beta_4 \vee \gamma_4 \\ \hline \end{array} \end{array}$$

These methods hold similarly for other A and k .

Theorem

\mathfrak{B}_A^k is a complete boolean algebra, for every set A and $k \geq 1$.

^a In our notation we write $f = \left[\begin{array}{c} X \longrightarrow Y \\ x \mapsto f(x) \end{array} \right]$ for a function $f : X \longrightarrow Y$ that maps each $x \in X$ to a well-defined $f(x) \in Y$.

Figure 3: Hyper-propositional logic

Formulas

For every set A and $k \in \mathbb{N}$ we define \mathbb{F}_A^k the (hyper-propositional) formulas of carrier A and degree k recursively as follows

- (i) If $k = 0$ then $\mathbb{F}_A^0 := A$.
(ii) If $k > 0$ then \mathbb{F}_A^k comprises the following expressions:

$$\left. \begin{array}{l} \diamond_k \sigma \quad (\text{diamond}) \\ \square_k \sigma \quad (\text{box}) \end{array} \right\} \text{ for all } \sigma \in \mathbb{F}_A^{k-1} \quad \left. \begin{array}{l} \neg_k \varphi \quad (\text{negation}) \\ [\varphi_1 \wedge_k \dots \wedge_k \varphi_n] \quad (\text{conjunction}) \\ [\varphi_1 \vee_k \dots \vee_k \varphi_n] \quad (\text{disjunction}) \end{array} \right\} \text{ for all } \varphi, \varphi_1, \dots, \varphi_n \in \mathbb{F}_A^k$$

We write $[\wedge_k]$ and $[\vee_k]$ for nullary, and $[\wedge_k \varphi_1]$ and $[\vee_k \varphi_1]$ for unary conjunctions and disjunctions, respectively.

Super-models and model classes

For every class A and every natural number $k \in \mathbb{N}$ we define the (super-) model function

$$\mathbf{mod}_A^{k+1} : \mathbb{F}_A^k \longrightarrow \mathbb{B}_A^k \longrightarrow \mathbb{B}$$

where $\mathbf{mod}_A^{k+1}(\varphi)(\Omega)$ is defined, for each $\varphi \in \mathbb{F}_A^k$ and $\Omega \in \mathbb{B}_A^k$, by induction on k as follows:

- (i) If $k = 0$ then $\varphi \in \mathbb{F}_A^0 = A$ and $\Omega \in \mathbb{B}_A^0 = A$ and

$$\mathbf{mod}_A^1(\varphi)(\Omega) := \begin{cases} \mathbf{1} & \text{if } \varphi = \Omega \\ \mathbf{0} & \text{else} \end{cases}$$

- (ii) If $k > 0$, we define by structural induction on the form of φ as follows:

$$\begin{aligned} \mathbf{mod}_A^{k+1} \left(\diamond_k \sigma \right) (\Omega) &:= \bigvee \{ \mathbf{mod}_A^k(\sigma)(\omega) \mid \omega \in \mathbb{B}_A^{k-1}, \Omega(\omega) = \mathbf{1} \} \\ \mathbf{mod}_A^{k+1} \left(\square_k \sigma \right) (\Omega) &:= \bigwedge \{ \mathbf{mod}_A^k(\sigma)(\omega) \mid \omega \in \mathbb{B}_A^{k-1}, \Omega(\omega) = \mathbf{1} \} \\ \mathbf{mod}_A^{k+1} \left(\neg_k \varphi \right) (\Omega) &:= \neg \mathbf{mod}_A^{k+1}(\varphi)(\Omega) \\ \mathbf{mod}_A^{k+1} \left([\varphi_1 \wedge_k \dots \wedge_k \varphi_n] \right) (\Omega) &:= \bigwedge \{ \mathbf{mod}_A^{k+1}(\varphi_1)(\Omega), \dots, \mathbf{mod}_A^{k+1}(\varphi_n)(\Omega) \} \\ \mathbf{mod}_A^{k+1} \left([\varphi_1 \vee_k \dots \vee_k \varphi_n] \right) (\Omega) &:= \bigvee \{ \mathbf{mod}_A^{k+1}(\varphi_1)(\Omega), \dots, \mathbf{mod}_A^{k+1}(\varphi_n)(\Omega) \} \end{aligned}$$

Furthermore:

- (α) $\mathbf{mod}_A^{k+1}(\varphi)(\Omega) \in \mathbb{B}$ is the so-called **truth value** of φ and (the **interpretation**) Ω
(β) If $\mathbf{mod}_A^{k+1}(\varphi)(\Omega) = \mathbf{1}$ we say that “ Ω is a **model** for φ ” or “ Ω **satisfies** φ ”, and this is also expressed by writing $\Omega \models \varphi$.
(γ) Accordingly and for each given $\varphi \in \mathbb{F}_A^k$, its **model class** is a subset of \mathbb{B}_A^k , defined by

$$\mathbf{Mod}_A^k(\varphi) := \{ \Omega \in \mathbb{B}_A^k \mid \mathbf{mod}_A^{k+1}(\varphi)(\Omega) = \mathbf{1} \}$$

- (δ) Note, that for each $k \in \mathbb{N}$, $\mathbf{mod}_A^{k+1} : \mathbb{F}_A^k \longrightarrow \mathbb{B}_A^{k+1}$, because $\mathbb{B}_A^{k+1} = (\mathbb{B}_A^k \longrightarrow \mathbb{B})$ (hence the superscript “ $k+1$ ” in “ \mathbf{mod}_A^{k+1} ”). We call $\mathbf{mod}_A^{k+1}(\varphi) \in \mathbb{B}_A^{k+1}$ the **super-model** or **truth table** of $\varphi \in \mathbb{F}_A^k$.

Subvalence and equivalence

Given A and k , we define two relations on \mathbb{F}_A^k . For all $\varphi, \psi \in \mathbb{F}_A^k$ let

$$\begin{array}{ll} \varphi \Rightarrow_A^k \psi & \text{iff } \forall \Omega \in \mathbb{B}_A^k . (\Omega \models \varphi \text{ implies } \Omega \models \psi) \\ & \text{iff } \mathbf{Mod}_A^k(\varphi) \subseteq \mathbf{Mod}_A^k(\psi) \\ & \text{iff } \mathbf{mod}_A^{k+1}(\varphi) \sqsubseteq_A^{k+1} \mathbf{mod}_A^{k+1}(\psi) \end{array} \quad \begin{array}{ll} \varphi \Leftrightarrow_A^k \psi & \text{iff } \forall \Omega \in \mathbb{B}_A^k . (\Omega \models \varphi \text{ iff } \Omega \models \psi) \\ & \text{iff } \mathbf{Mod}_A^k(\varphi) = \mathbf{Mod}_A^k(\psi) \\ & \text{iff } \mathbf{mod}_A^{k+1}(\varphi) = \mathbf{mod}_A^{k+1}(\psi) \end{array}$$

If $\varphi \Rightarrow_A^k \psi$ then we say that “ φ is **subvalent** to ψ ” or “ φ **implies** ψ ” or “ φ **entails** ψ ” or “ ψ is a **consequence** of φ ”. And $\varphi \Leftrightarrow_A^k \psi$ is read as “ φ and ψ are **equivalent**”.

The quasi-boolean lattice of formulas

$\mathfrak{F}_A^k := \langle \mathbb{F}_A^k, \Rightarrow_A^k, \Leftrightarrow_A^k, \mathbf{f}^k, \mathbf{t}^k, \wedge^k, \vee^k, \neg^k \rangle$ is the **default formula algebra** of A and k , where for all $\varphi, \psi \in \mathbb{F}_A^k$

$$\mathbf{f}^k := [\vee_k] \quad \mathbf{t}^k := [\wedge_k] \quad \varphi \wedge^k \psi := [\varphi \wedge_k \psi] \quad \varphi \vee^k \psi := [\varphi \vee_k \psi] \quad \neg^k \varphi := \neg_k \varphi$$

Theorem

\mathfrak{F}_A^k is a quasi-boolean algebra, for every A and $k \geq 1$.

Theorem

$\mathbf{mod}_A^{k+1} : \mathfrak{F}_A^k \hookrightarrow \mathfrak{B}_A^{k+1}$, i.e. $\mathbf{mod}_A^{k+1} : \mathbb{F}_A^k \longrightarrow \mathbb{B}_A^{k+1}$ is an embedding of \mathfrak{F}_A^k into \mathfrak{B}_A^{k+1} , for all A and $k \geq 1$.