Summarized overview of set field logic and its embedding into hyper–propositional logic

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Figure 1: Bit values and their algebra

Bit values

$\mathbb{B} := \{\mathbf{0}, \mathbf{1}\}$ is the bit value class, where $\mathbf{0}$ is the zero bit and $\mathbf{1}$ the unit bit.

Bit value algebra

$\mathfrak{B} := \langle \mathbb{B}, \leq, \mathbf{0}, \mathbf{1}, \wedge, \vee, \bigwedge, \bigvee, \text{-} \rangle$ is the bit value algebra, where

$\beta_1 \leq \beta_2$ iff $\beta_1 = \mathbf{0}$ or $\beta_2 = \mathbf{1}$

$$\beta_1 \wedge \beta_2 := \bigwedge \{\beta_1, \beta_2\} \qquad \bigwedge \mathcal{B} := \begin{cases} \mathbf{0} & \text{if } \mathbf{0} \in \mathcal{B} \\ \mathbf{1} & \text{else} \end{cases}$$

$$\beta_1 \vee \beta_2 := \bigvee \{\beta_1, \beta_2\} \qquad \bigvee \mathcal{B} := \begin{cases} \mathbf{1} & \text{if } \mathbf{1} \in \mathcal{B} \\ \mathbf{0} & \text{else} \end{cases}$$

$$\text{-}\beta := \begin{cases} \mathbf{0} & \text{if } \beta = \mathbf{1} \\ \mathbf{1} & \text{else} \end{cases}$$

for all $\beta, \beta_1, \beta_2 \in \mathbb{B}$ and $\mathcal{B} \subseteq \mathbb{B}$.

We also write, for all $\beta_1, \ldots, \beta_n \in \mathbb{B}$ with $n \geq 0$,

$$\bigwedge_{i=1}^{n} \beta_i \text{ for } \bigwedge \{\beta_1, \ldots, \beta_n\} \qquad \text{and} \qquad \bigvee_{i=1}^{n} \beta_i \text{ for } \bigvee \{\beta_1, \ldots, \beta_n\}$$

Theorem

$\mathfrak{B}$ is a complete boolean algebra.

Figure 2: Bit tables and their algebras

Bit tables

For every set A and each natural number k we define

$$\mathbb{B}^k_A \quad := \quad \begin{cases} A & \text{if } k = 0 \\ \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} & \text{if } k > 0 \end{cases}$$

the bit table set of carrier A and degree k.

In our default notation for functions[a], each bit table $\Omega \in \mathbb{B}^k_A$ with $k \geq 1$ is then given by

$$\Omega = \begin{bmatrix} \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} \\ \omega \mapsto \Omega(\omega) \end{bmatrix}$$

Similar to geometry, bit tables of small degree $k = 0, 1, 2, 3$ are also called *bit points*, *bit lines*, *bit squares* and *bit cubes*, respectively. In traditional propositional logic, bit squares are also known as *truth tables*.

Bit table diagrams

If both A and k are finite, we can represent each $\Omega \in \mathbb{B}^k_A$ by its bit table diagram. For example

(1) If $A = \{a, b\}$ and $k = 1$ then

$$\begin{array}{|c|c|c|} \hline a & b & c \\ \hline \beta_1 & \beta_2 & \beta_3 \\ \hline \end{array} \quad := \quad \begin{bmatrix} A \longrightarrow \mathbb{B} \\ \begin{bmatrix} a \mapsto \beta_1 \\ b \mapsto \beta_2 \\ c \mapsto \beta_3 \end{bmatrix} \end{bmatrix}$$

(2) If $A = \{a, b\}$ and $k = 2$ then

$$\begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & \beta_3 \\ 1 & 1 & \beta_4 \\ \hline \end{array} \quad := \quad \begin{bmatrix} \mathbb{B}^1_A \longrightarrow \mathbb{B} \\ \begin{bmatrix} \begin{array}{|c|c|} \hline a & b \\ \hline 0 & 0 \\ \hline \end{array} \mapsto \beta_1 \\ \begin{array}{|c|c|} \hline a & b \\ \hline 1 & 0 \\ \hline \end{array} \mapsto \beta_2 \\ \begin{array}{|c|c|} \hline a & b \\ \hline 0 & 1 \\ \hline \end{array} \mapsto \beta_3 \\ \begin{array}{|c|c|} \hline a & b \\ \hline 1 & 1 \\ \hline \end{array} \mapsto \beta_4 \end{bmatrix} \end{bmatrix}$$

(3) If $A = \{a\}$ and $k = 3$ then

$$\begin{array}{|c||c|c|c|c|} \hline a & & & & \\ \hline 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \hline \end{array} \quad := \quad \begin{bmatrix} \mathbb{B}^2_A \longrightarrow \mathbb{B} \\ \begin{bmatrix} \begin{array}{|c||c|} \hline a & \\ \hline 0 & 0 \\ 1 & 0 \\ \hline \end{array} \mapsto \beta_1 \\ \begin{array}{|c||c|} \hline a & \\ \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array} \mapsto \beta_2 \\ \begin{array}{|c||c|} \hline a & \\ \hline 0 & 0 \\ 1 & 1 \\ \hline \end{array} \mapsto \beta_3 \\ \begin{array}{|c||c|} \hline a & \\ \hline 0 & 1 \\ 1 & 1 \\ \hline \end{array} \mapsto \beta_4 \end{bmatrix} \end{bmatrix}$$

Bit table algebras

$\mathfrak{B}^k_A := \langle \mathbb{B}^k_A, \sqsubseteq^k_A, \bot^k_A, \top^k_A, \sqcap^k_A, \sqcup^k_A, \prod^k_A, \coprod^k_A, \neg^k_A \rangle$ is the bit table algebra, for each set A and $k \geq 1$, where

$$\Omega \sqsubseteq^k_A \Omega' \quad \text{iff } \Omega(\omega) \leq \Omega'(\omega) \text{ for all } \omega \in \mathbb{B}^{k-1}_A$$

$$\Omega \sqcap^k_A \Omega' := \begin{bmatrix} \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} \\ \omega \mapsto \Omega(\omega) \wedge \Omega'(\omega) \end{bmatrix} \qquad \Omega \sqcup^k_A \Omega' := \begin{bmatrix} \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} \\ \omega \mapsto \Omega(\omega) \vee \Omega'(\omega) \end{bmatrix} \qquad \neg^k_A \Omega := \begin{bmatrix} \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} \\ \omega \mapsto \text{-}\Omega(\omega) \end{bmatrix}$$

$$\bot^k_A := \begin{bmatrix} \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} \\ \omega \mapsto \mathbf{0} \end{bmatrix} \qquad \top^k_A := \begin{bmatrix} \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} \\ \omega \mapsto \mathbf{1} \end{bmatrix} \qquad \prod^k_A \Gamma := \begin{bmatrix} \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} \\ \omega \mapsto \bigwedge\{\Omega(\omega) \mid \Omega \in \Gamma\} \end{bmatrix} \qquad \coprod^k_A \Gamma := \begin{bmatrix} \mathbb{B}^{k-1}_A \longrightarrow \mathbb{B} \\ \omega \mapsto \bigvee\{\Omega(\omega) \mid \Omega \in \Gamma\} \end{bmatrix}$$

for all $\Omega, \Omega' \in \mathbb{B}^k_A$ and $\Gamma \subseteq \mathbb{B}^k_A$.

Using bit table diagrams and taking $A = \{a, b\}$ and $k = 2$ for example, the operations are

$$\begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & \beta_3 \\ 1 & 1 & \beta_4 \\ \hline \end{array} \sqsubseteq^2_A \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \gamma_1 \\ 1 & 0 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 1 & 1 & \gamma_4 \\ \hline \end{array} \text{ iff } \begin{pmatrix} \beta_1 \leq \delta_1 \text{ and} \\ \beta_2 \leq \delta_2 \text{ and} \\ \beta_3 \leq \delta_3 \text{ and} \\ \beta_4 \leq \delta_4 \end{pmatrix} \qquad \bot^2_A = \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ \hline \end{array} \qquad \top^2_A = \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ \hline \end{array}$$

$$\neg^2_A \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & \beta_3 \\ 1 & 1 & \beta_4 \\ \hline \end{array} = \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \text{-}\beta_1 \\ 1 & 0 & \text{-}\beta_2 \\ 0 & 1 & \text{-}\beta_3 \\ 1 & 1 & \text{-}\beta_4 \\ \hline \end{array} \qquad \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & \beta_3 \\ 1 & 1 & \beta_4 \\ \hline \end{array} \sqcap^2_A \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \gamma_1 \\ 1 & 0 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 1 & 1 & \gamma_4 \\ \hline \end{array} = \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \wedge \gamma_1 \\ 1 & 0 & \beta_2 \wedge \gamma_2 \\ 0 & 1 & \beta_3 \wedge \gamma_3 \\ 1 & 1 & \beta_4 \wedge \gamma_4 \\ \hline \end{array} \qquad \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \\ 1 & 0 & \beta_2 \\ 0 & 1 & \beta_3 \\ 1 & 1 & \beta_4 \\ \hline \end{array} \sqcup^2_A \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \gamma_1 \\ 1 & 0 & \gamma_2 \\ 0 & 1 & \gamma_3 \\ 1 & 1 & \gamma_4 \\ \hline \end{array} = \begin{array}{|c|c||c|} \hline a & b & \\ \hline 0 & 0 & \beta_1 \vee \gamma_1 \\ 1 & 0 & \beta_2 \vee \gamma_2 \\ 0 & 1 & \beta_3 \vee \gamma_3 \\ 1 & 1 & \beta_4 \vee \gamma_4 \\ \hline \end{array}$$

These methods hold similarly for other A and k.

Theorem

$\mathfrak{B}^k_A$ is a complete boolean algebra, for every set A and $k \geq 1$.

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[a] In our notation we write $f = \begin{bmatrix} X \longrightarrow Y \\ x \mapsto f(x) \end{bmatrix}$ for a function $f : X \longrightarrow Y$ that maps each $x \in X$ to a well–defined $f(x) \in Y$.

Figure 3: Hyper–propositional logic

Formulas

For every set A and $k \in \mathbb{N}$ we define $\mathbb{F}^k_A$ the (hyper–propositional) formulas of carrier A and degree k recursively as follows

(i) If $k = 0$ then $\mathbb{F}^0_A := A$.

(ii) If $k > 0$ then $\mathbb{F}^k_A$ comprises the following expressions:

- $\underset{k}{\Diamond}\sigma$ (diamond) and $\underset{k}{\Box}\sigma$ (box), for all $\sigma \in \mathbb{F}^{k-1}_A$
- $\underset{k}{\neg}\varphi$ (negation), $[\varphi_1 \underset{k}{\wedge} \dots \underset{k}{\wedge} \varphi_n]$ (conjunction), $[\varphi_1 \underset{k}{\vee} \dots \underset{k}{\vee} \varphi_n]$ (disjunction), for all $\varphi, \varphi_1, \dots, \varphi_n \in \mathbb{F}^k_A$

We write $[\underset{k}{\wedge}]$ and $[\underset{k}{\vee}]$ for nullary, and $[\underset{k}{\wedge}\varphi_1]$ and $[\underset{k}{\vee}\varphi_1]$ for unary conjunctions and disjunctions, respectively.

Super–models and model classes

For every class A and every natural number $k \in \mathbb{N}$ we define the (super–) model function

$$\mathbf{mod}^{k+1}_A : \mathbb{F}^k_A \longrightarrow \mathbb{B}^k_A \longrightarrow \mathbb{B}$$

where $\mathbf{mod}^{k+1}_A(\varphi)(\Omega)$ is defined, for each $\varphi \in \mathbb{F}^k_A$ and $\Omega \in \mathbb{B}^k_A$, by induction on k as follows:

(i) If $k = 0$ then $\varphi \in \mathbb{F}^0_A = A$ and $\Omega \in \mathbb{B}^0_A = A$ and

$$\mathbf{mod}^1_A(\varphi)(\Omega) := \begin{cases} \mathbf{1} & \text{if } \varphi = \Omega \\ \mathbf{0} & \text{else} \end{cases}$$

(ii) If $k > 0$, we define by structural induction on the form of φ as follows:

$$\begin{aligned}
\mathbf{mod}^{k+1}_A\left(\underset{k}{\Diamond}\sigma\right)(\Omega) &:= \bigvee\{\mathbf{mod}^k_A(\sigma)(\omega) \mid \omega \in \mathbb{B}^{k-1}_A, \Omega(\omega) = \mathbf{1}\} \\
\mathbf{mod}^{k+1}_A\left(\underset{k}{\Box}\sigma\right)(\Omega) &:= \bigwedge\{\mathbf{mod}^k_A(\sigma)(\omega) \mid \omega \in \mathbb{B}^{k-1}_A, \Omega(\omega) = \mathbf{1}\} \\
\mathbf{mod}^{k+1}_A\left(\underset{k}{\neg}\varphi\right)(\Omega) &:= \text{-}\mathbf{mod}^{k+1}_A(\varphi)(\Omega) \\
\mathbf{mod}^{k+1}_A\left([\varphi_1 \underset{k}{\wedge} \dots \underset{k}{\wedge} \varphi_n]\right)(\Omega) &:= \bigwedge\left\{\mathbf{mod}^{k+1}_A(\varphi_1)(\Omega), \dots, \mathbf{mod}^{k+1}_A(\varphi_n)(\Omega)\right\} \\
\mathbf{mod}^{k+1}_A\left([\varphi_1 \underset{k}{\vee} \dots \underset{k}{\vee} \varphi_n]\right)(\Omega) &:= \bigvee\left\{\mathbf{mod}^{k+1}_A(\varphi_1)(\Omega), \dots, \mathbf{mod}^{k+1}_A(\varphi_n)(\Omega)\right\}
\end{aligned}$$

Furthermore:

(α) $\mathbf{mod}^{k+1}_A(\varphi)(\Omega) \in \mathbb{B}$ is the so–called truth value of φ and (the interpretation) Ω

(β) If $\mathbf{mod}^{k+1}_A(\varphi)(\Omega) = \mathbf{1}$ we say that "Ω is a model for φ" or "Ω satisfies φ", and this is also expressed by writing $\Omega \models \varphi$.

(γ) Accordingly and for each given $\varphi \in \mathbb{F}^k_A$, its model class is a subset of $\mathbb{B}^k_A$, defined by

$$\mathbf{Mod}^k_A(\varphi) := \{\Omega \in \mathbb{B}^k_A \mid \mathbf{mod}^{k+1}_A(\varphi)(\Omega) = \mathbf{1}\}$$

(δ) Note, that for each $k \in \mathbb{N}$, $\mathbf{mod}^{k+1}_A : \mathbb{F}^k_A \longrightarrow \mathbb{B}^{k+1}_A$, because $\mathbb{B}^{k+1}_A = (\mathbb{B}^k_A \longrightarrow \mathbb{B})$ (hence the superscript "$k+1$" in "$\mathbf{mod}^{k+1}_A$"). We call $\mathbf{mod}^{k+1}_A(\varphi) \in \mathbb{B}^{k+1}_A$ the super–model or truth table of $\varphi \in \mathbb{F}^k_A$.

Subvalence and equivalence

Given A and k, we define two relations on $\mathbb{F}^k_A$. For all $\varphi, \psi \in \mathbb{F}^k_A$ let

$$\begin{aligned}
\varphi \Rightarrow^k_A \psi \quad &\text{iff} \quad \forall \Omega \in \mathbb{B}^k_A \,.\, (\Omega \models \varphi \text{ implies } \Omega \models \psi) \\
&\text{iff} \quad \mathbf{Mod}^k_A(\varphi) \subseteq \mathbf{Mod}^k_A(\psi) \\
&\text{iff} \quad \mathbf{mod}^{k+1}_A(\varphi) \sqsubseteq^{k+1}_A \mathbf{mod}^{k+1}_A(\psi)
\end{aligned}$$

$$\begin{aligned}
\varphi \Leftrightarrow^k_A \psi \quad &\text{iff} \quad \forall \Omega \in \mathbb{B}^k_A \,.\, (\Omega \models \varphi \text{ iff } \Omega \models \psi) \\
&\text{iff} \quad \mathbf{Mod}^k_A(\varphi) = \mathbf{Mod}^k_A(\psi) \\
&\text{iff} \quad \mathbf{mod}^{k+1}_A(\varphi) = \mathbf{mod}^{k+1}_A(\psi)
\end{aligned}$$

If $\varphi \Rightarrow^k_A \psi$ then we say that "φ is subvalent to ψ" or "φ implies ψ" or "φ entails ψ" or "ψ is a consequence of φ". And $\varphi \Leftrightarrow^k_A \psi$ is read as "φ and ψ are equivalent".

The quasi–boolean lattice of formulas

$\mathfrak{F}^k_A := \langle \mathbb{F}^k_A, \Rightarrow^k_A, \Leftrightarrow^k_A, \mathbf{f}^k, \mathbf{t}^k, \wedge^k, \vee^k, \neg^k \rangle$ is the default formula algebra of A and k, where for all $\varphi, \psi \in \mathbb{F}^k_A$

$\mathbf{f}^k := [\underset{k}{\vee}]$ $\qquad$ $\mathbf{t}^k := [\underset{k}{\wedge}]$ $\qquad$ $\varphi \wedge^k \psi := [\varphi \underset{k}{\wedge} \psi]$ $\qquad$ $\varphi \vee^k \psi := [\varphi \underset{k}{\vee} \psi]$ $\qquad$ $\neg^k \varphi := \underset{k}{\neg}\varphi$

Theorem

$\mathfrak{F}^k_A$ is a quasi–boolean algebra, for every A and $k \geq 1$.

Theorem

$\mathbf{mod}^{k+1}_A : \mathfrak{F}^k_A \hookrightarrow \mathfrak{B}^{k+1}_A$, i.e. $\mathbf{mod}^{k+1}_A : \mathbb{F}^k_A \longrightarrow \mathbb{B}^{k+1}_A$ is an embedding of $\mathfrak{F}^k_A$ into $\mathfrak{B}^{k+1}_A$, for all A and $k \geq 1$.

Figure 4: Set field logic

Set expressions

For every class A we define

(i) A itself is called the set variable class

(ii) $\mathbf{Stm}(A)$, the set term class of A, comprises

a	for each $a \in A$	(set symbol)
empty		(empty set symbol)
full		(full set symbol)
$[\sigma \setminus \vartheta]$	for all $\sigma, \vartheta \in \mathbf{Stm}(A)$	(difference)
$[\sigma_1 \cap \ldots \cap \sigma_n]$	for all $\sigma_1, \ldots, \sigma_n \in \mathbf{Stm}(A)$	(intersection)
$[\sigma_1 \cup \ldots \cup \sigma_n]$	for all $\sigma_1, \ldots, \sigma_n \in \mathbf{Stm}(A)$	(union)

(iii) $\mathbf{Sfm}(A)$, the set formula class of A, comprises

$[\sigma \subseteq \vartheta]$	for all $\sigma, \vartheta \in \mathbf{Stm}(A)$	(inclusion)
false		(false symbol)
true		(true symbol)
$\neg\varphi$	for all $\varphi \in \mathbf{Sfm}(A)$	(negation)
$[\varphi_1 \wedge \ldots \wedge \varphi_n]$	for all $\varphi_1, \ldots, \varphi_n \in \mathbf{Sfm}(A)$	(conjunction)
$[\varphi_1 \vee \ldots \vee \varphi_n]$	for all $\varphi_1, \ldots, \varphi_n \in \mathbf{Sfm}(A)$	(disjunction)

Set field interpretations

For every class A, a set field interpretation of A is given by a function $\mathfrak{I} : A \longrightarrow \mathbf{P}(C)$, where the set C is the so–called carrier of $\mathfrak{I}$.

$\mathbf{Sfint}(A)$ denotes the class of all such set field interpretations on A.

Each $\mathfrak{I} \in \mathbf{Sfint}(A)$ induces two more functions:

(i) $\mathbf{set}_{\mathfrak{I}} : \mathbf{Stm}(A) \longrightarrow \mathbf{P}(C)$, the set function of $\mathfrak{I}$, that returns a subset $\mathbf{set}_{\mathfrak{I}}(\sigma)$ of C for every set term σ, defined by

$$\begin{aligned}
\mathbf{set}_{\mathfrak{I}}(a) &:= \mathfrak{I}(a)\\
\mathbf{set}_{\mathfrak{I}}(\mathbf{empty}) &:= \emptyset\\
\mathbf{set}_{\mathfrak{I}}(\mathbf{full}) &:= C\\
\mathbf{set}_{\mathfrak{I}}([\sigma \setminus \vartheta]) &:= \mathbf{set}_{\mathfrak{I}}(\sigma) \setminus \mathbf{set}_{\mathfrak{I}}(\vartheta)\\
\mathbf{set}_{\mathfrak{I}}([\sigma_1 \cap \ldots \cap \sigma_n]) &:= \mathbf{set}_{\mathfrak{I}}(\sigma_1) \cap \ldots \cap \mathbf{set}_{\mathfrak{I}}(\sigma_n)\\
\mathbf{set}_{\mathfrak{I}}([\sigma_1 \cup \ldots \cup \sigma_n]) &:= \mathbf{set}_{\mathfrak{I}}(\sigma_1) \cup \ldots \cup \mathbf{set}_{\mathfrak{I}}(\sigma_n)
\end{aligned}$$

(ii) $\mathbf{truth}_{\mathfrak{I}} : \mathbf{Sfm}(A) \longrightarrow \mathbb{B}$, the truth value function of $\mathfrak{I}$, which returns a bit value $\mathbf{truth}_{\mathfrak{I}}(\varphi)$ for every set formula φ, defined by

$$\begin{aligned}
\mathbf{truth}_{\mathfrak{I}}([\sigma \subseteq \vartheta]) &:= \begin{cases}\mathbf{1} & \text{if } \mathbf{set}_{\mathfrak{I}}(\sigma) \subseteq \mathbf{set}_{\mathfrak{I}}(\vartheta)\\ \mathbf{0} & \text{else}\end{cases}\\
\mathbf{truth}_{\mathfrak{I}}(\mathbf{false}) &:= \mathbf{0}\\
\mathbf{truth}_{\mathfrak{I}}(\mathbf{true}) &:= \mathbf{1}\\
\mathbf{truth}_{\mathfrak{I}}(\neg\varphi) &:= \text{-}\mathbf{truth}_{\mathfrak{I}}(\varphi)\\
\mathbf{truth}_{\mathfrak{I}}([\varphi_1 \wedge \ldots \wedge \varphi_n]) &:= \mathbf{truth}_{\mathfrak{I}}(\varphi_1) \wedge \ldots \wedge \mathbf{truth}_{\mathfrak{I}}(\varphi_n)\\
\mathbf{truth}_{\mathfrak{I}}([\varphi_1 \vee \ldots \vee \varphi_n]) &:= \mathbf{truth}_{\mathfrak{I}}(\varphi_1) \vee \ldots \vee \mathbf{truth}_{\mathfrak{I}}(\varphi_n)
\end{aligned}$$

Set term and set formula algebra

For every given A we define

(i) $\mathfrak{Stm}(A) := \langle \mathbf{Stm}(A), \sqsubseteq, \equiv, \bot, \top, \sqcap, \sqcup, \neg \rangle$, the default set term algebra of A, where for all $\sigma, \vartheta \in \mathbf{Stm}(A)$

$\sigma \sqsubseteq \vartheta$ iff $\forall \mathfrak{I} \in \mathbf{Sfint}(A) \, . \, \mathbf{set}_{\mathfrak{I}}(\sigma) \subseteq \mathbf{set}_{\mathfrak{I}}(\vartheta)$ $\quad$ $\sigma \equiv \vartheta$ iff $\forall \mathfrak{I} \in \mathbf{Sfint}(A) \, . \, \mathbf{set}_{\mathfrak{I}}(\sigma) = \mathbf{set}_{\mathfrak{I}}(\vartheta)$

$\bot := \mathbf{empty}$ $\quad$ $\top := \mathbf{full}$ $\quad$ $\sigma \sqcap \vartheta := [\sigma \cap \vartheta]$ $\quad$ $\sigma \sqcup \vartheta := [\sigma \cup \vartheta]$ $\quad$ $\neg\sigma := [\mathbf{full} \setminus \sigma]$

(ii) $\mathfrak{Sfm}(A) := \langle \mathbf{Sfm}(A), \sqsubseteq, \equiv, \bot, \top, \sqcap, \sqcup, \neg \rangle$, the default set formula algebra of A, where for all $\varphi, \psi, \in \mathbf{Sfm}(A)$

$\varphi \sqsubseteq \psi$ iff $\forall \mathfrak{I} \in \mathbf{Sfint}(A) \, . \, \mathbf{truth}_{\mathfrak{I}}(\varphi) \leq \mathbf{truth}_{\mathfrak{I}}(\psi)$ $\quad$ $\varphi \equiv \psi$ iff $\forall \mathfrak{I} \in \mathbf{Sfint}(A) \, . \, \mathbf{truth}_{\mathfrak{I}}(\varphi) = \mathbf{truth}_{\mathfrak{I}}(\psi)$

$\bot := \mathbf{false}$ $\quad$ $\top := \mathbf{true}$ $\quad$ $\varphi \sqcap \psi := [\varphi \wedge \psi]$ $\quad$ $\varphi \sqcup \psi := [\varphi \vee \psi]$ $\quad$ $\neg\varphi := \neg\varphi$

Theorem

For every A holds:

(i) $\mathfrak{Stm}(A)$ is a quasi–boolean algebra

(ii) $\mathfrak{Sfm}(A)$ is a quasi–boolean algebra

Figure 5: Embedding set field logic into hyper–propositional logic

Let A be an arbitrary class.

Theorem (The embedding of set field logic into hyper–propositional logic)

(i) $\dot{\mathbf{f}} : \mathfrak{Stm}(A) \hookrightarrow \mathfrak{F}^1_A$, i.e. $\dot{\mathbf{f}}$ is an embedding of $\mathfrak{Stm}(A)$ into $\mathfrak{F}^1_A$, where

$$\dot{\mathbf{f}} \quad := \quad \left[\begin{array}{c} \mathbf{Stm}(A) \longrightarrow \mathbb{F}^1_A \\ \left[\begin{array}{r} a \mapsto \underset{1}{\Diamond} a \\ \mathbf{empty} \mapsto [\underset{1}{\vee}] \\ \mathbf{full} \mapsto [\underset{1}{\wedge}] \\ [\,\sigma \setminus \vartheta\,] \mapsto [\,\dot{\mathbf{f}}(\sigma) \underset{1}{\wedge} \underset{1}{\neg} \dot{\mathbf{f}}(\vartheta)\,] \\ [\,\sigma_1 \cap \ldots \cap \sigma_n\,] \mapsto [\,\dot{\mathbf{f}}(\sigma_1) \underset{1}{\wedge} \ldots \underset{1}{\wedge} \dot{\mathbf{f}}(\sigma_n)\,] \\ [\,\sigma_1 \cup \ldots \cup \sigma_n\,] \mapsto [\,\dot{\mathbf{f}}(\sigma_1) \underset{1}{\vee} \ldots \underset{1}{\vee} \dot{\mathbf{f}}(\sigma_n)\,] \end{array}\right] \end{array}\right]$$

(ii) $\ddot{\mathbf{f}} : \mathfrak{Sfm}(A) \hookrightarrow \mathfrak{F}^2_A$, i.e. $\ddot{\mathbf{f}}$ is an embedding of $\mathfrak{Sfm}(A)$ into $\mathfrak{F}^2_A$, where

$$\ddot{\mathbf{f}} \quad := \quad \left[\begin{array}{c} \mathbf{Sfm}(A) \longrightarrow \mathbb{F}^2_A \\ \left[\begin{array}{r} [\,\sigma \subseteq \vartheta\,] \mapsto \underset{2}{\Box}[\,\underset{1}{\neg} \dot{\mathbf{f}}(\sigma) \underset{1}{\vee} \dot{\mathbf{f}}(\vartheta)\,] \\ \mathbf{false} \mapsto [\underset{2}{\vee}] \\ \mathbf{true} \mapsto [\underset{2}{\wedge}] \\ \neg\varphi \mapsto \underset{2}{\neg} \ddot{\mathbf{f}}(\varphi) \\ [\,\varphi_1 \wedge \ldots \wedge \varphi_n\,] \mapsto [\,\ddot{\mathbf{f}}(\varphi_1) \underset{2}{\wedge} \ldots \underset{2}{\wedge} \ddot{\mathbf{f}}(\varphi_n)\,] \\ [\,\varphi_1 \vee \ldots \vee \varphi_n\,] \mapsto [\,\ddot{\mathbf{f}}(\varphi_1) \underset{2}{\vee} \ldots \underset{2}{\vee} \ddot{\mathbf{f}}(\varphi_n)\,] \end{array}\right] \end{array}\right]$$