The Algebra of Worlds

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Contents

1 Introduction

We are going to define the concept of a *world algebra* on a given set $\mathcal{A}$. One such *world* is a function $(A \longrightarrow \mathcal{B}) \longrightarrow \mathcal{B}$, where $A \subseteq \mathcal{A}$ and $\mathcal{B}$ is the set of the two boolean or bit values.

Such a world algebra is a standard model of a *theory algebra*. Another example of such a theory algebra is the set of (boolean or theory) formulas on $\mathcal{A}$, equipped with the according operations and relations. There is a kind of isomorphism between the world and the formula algebra: each formula represents a unique world and every (finite) world can be represented by a formula. Worlds are — as the term suggests — very "meaningful" and intuitive. Formulas on the other hand have the advantage that they can display certain properties of the according world and that they are much more suitable as data structures for computer programs which implement theory algebras. So, there is a productive interplay between formulas and worlds.

2 Worlds and their representation

2.1 Worlds

$\mathcal{A}$ denotes a pre–defined set of **atoms**. In our case $\mathcal{A}$ is made of all non–empty strings of letters, digits and the understroke.

There is a strict linear order relation $<$ defined on $\mathcal{A}$.

Because $\mathcal{A}$ is linearly ordered by $<$, every finite atom set A of n atoms has a unique representation as an ordered atom list $[\,\alpha_1\ \alpha_2\ \ldots\ \alpha_n\,]$, with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. There is a unique bijection between the finite subsets of $\mathcal{A}$ and the set of all ordered atom lists on $\mathcal{A}$. So the standard set relations $\in$, $\subseteq$ and operations $\cap$, $\cup$, $\setminus$ on sets are well–defined on ordered atom lists as well, where the role of the empty set $\emptyset$ is taken by the **empty list** $[\,]$.

$\mathcal{B} := \{?, !\}$ denotes the set of bitvalues. It has exactly two elements:
- ? the **zero bit** (also called **false** and written as **0** or **f** or $\perp$)
- ! the **unit bit** (also called **true** and written as **1** or **t** or $\top$)

$\mathcal{B}$ is linearly ordered by defining for all $\beta, \beta' \in \mathcal{B}$

$$\beta < \beta' \quad :\text{iff} \quad \beta = ? \text{ and } \beta' = !$$

If A is a finite set or ordered list of atoms, a (binary) **valuation of** A is a function $\omega : A \longrightarrow \mathcal{B}$. If A is the ordered list $[\,\alpha_1\ \ldots\ \alpha_n\,]$, then ω is also written as

$$[\,\alpha_1/\omega(\alpha_1)\ \ldots\ \alpha_n/\omega(\alpha_n)\,]$$

For example $\omega = [\,a/?\ b/!\ c/!\,]$ is a valuation of $A = [\,a\ b\ c\,]$.

We assume that there is a strict linear order $<$ defined on the set of all the valuations of A. For example, for $A = [\,a\ b\,]$ we put

$$[\,a/?\ b/?\,] < [\,a/!\ b/?\,] < [\,a/?\ b/!\,] < [\,a/!\ b/!\,]$$

$Val(A)$ denotes the set or ordered list of all the valuations of A.

For example, for $A = [\,a\ b\,]$

$$Val(A) = [\,[\,a/?\ b/?\,]\ [\,a/!\ b/?\,]\ [\,a/?\ b/!\,]\ [\,a/!\ b/!\,]\,]$$

If A contains n different atoms, then $Val(A)$ has 2^n different valuations of A.

Note that $Val([\,]) = [[\,]]$, i.e. the empty list $[\,]$ of atoms has exactly one valuation, also written $[\,]$, which is the **empty valuation**.

A (**finite binary**) **world** τ (**on** A) is defined by
- A finite set or ordered list A of atoms.
- A function $Val(A) \longrightarrow \mathcal{B}$.

In other words, τ is a function $\tau : (A \longrightarrow \mathcal{B}) \longrightarrow \mathcal{B}$

A valuation ω in a world τ is also said to be a
- **possible state in** τ, if $\tau(\omega) = !$
- **impossible state in** τ, if $\tau(\omega) = ?$

2.2 Special properties of worlds

A world τ on A is said to be

- **valid** or a **tautology** if every valuation is a possible state, i.e. if $\tau(\omega) = !$ for all $\omega \in Val(A)$
- **satisfiable** if at least one valuation is a possible state,
- **contradictory** or a **contradiction** if every valuation is an impossible state.

2.3 Atom lists of worlds

If τ is a world on an ordered atoms list A, then

$$@\tau := A$$

is **the (ordered) atom list of** τ.

2.4 Tables

If τ is a world on A, the **table** of τ is uniquely defined by

$$\begin{array}{r|l} A & \\ \hline \omega_0 & \tau(\omega_0) \\ \omega_1 & \tau(\omega_1) \\ \vdots & \vdots \\ \omega_{2^n-1} & \tau(\omega_{2^n-1}) \end{array} \qquad \text{or} \qquad \begin{array}{ccc|c} \alpha_1 & \dots & \alpha_n & \\ \hline \omega_0(\alpha_1) & \dots & \omega_0(\alpha_n) & \tau(\omega_0) \\ \omega_1(\alpha_1) & \dots & \omega_1(\alpha_n) & \tau(\omega_1) \\ \vdots & & \vdots & \vdots \\ \omega_{2^n-1}(\alpha_1) & \dots & \omega_{2^n-1}(\alpha_n) & \tau(\omega_{2^n-1}) \end{array}$$

where $[\,\alpha_1 \,\dots\, \alpha_n\,] := A$ and $[\,\omega_0\,\omega_1\,\dots\,\omega_{2^n-1}\,] := Val(A)$.

For example, let $A := [\,hot\ rain\ snow\,]$ be an orderd list of three atoms. A world τ on A is represented by the table

hot	*rain*	*snow*	
?	?	?	!
!	?	?	!
?	!	?	?
!	!	?	!
?	?	!	!
!	?	!	?
?	!	!	?
!	!	!	?

A possible state in w is given by, say $[\,hot/!\ rain/?\ snow/?\,]$. In other word, the state "it is hot, not raining, and not snowing" is a possible state in w, while for example "it is hot, it does not rain, and it snows" is an impossible state in w.

2.5 Double tables

Let $\omega_1 : A_1 \longrightarrow \mathcal{B}$ and $\omega_2 : A_2 \longrightarrow \mathcal{B}$ be two valuations on disjunct atom sets, i.e. $A_1 \cap A_2$ is empty. Then

$$\omega_1 \cdot \omega_2 : A_1 \cup A_2 \longrightarrow \mathcal{B} \quad \text{with} \quad \omega_1 \cdot \omega_2(\alpha) := \begin{cases} \omega_1(\alpha) & \text{if } \alpha \in A_1 \\ \omega_2(\alpha) & \text{if } \alpha \in A_2 \end{cases}$$

For example

$$[\,a/!\ c/!\,] \cdot [\,b/?\ d/?\,] = [\,a/!\ b/?\ c/!\ d/?\,]$$

Let τ be a world on an atom list A. We can split A in two atom lists: a left atom list A_L and an upper atom list A_U. We can then represent τ by its **double table** on A_L and A_U, which has the form

$$
\begin{array}{c|ccc|c}
 & \omega''_0 & \dots & \omega''_{2^u-1} & A_U \\
\hline
\omega'_0 & \tau(\omega'_0 \cdot \omega''_0) & \dots & \tau(\omega'_0 \cdot \omega''_{2^u-1}) & \\
\vdots & \vdots & & \vdots & \\
\omega'_{2^l-1} & \tau(\omega'_{2^l-1} \cdot \omega''_0) & \dots & \tau(\omega'_{2^l-1} \cdot \omega''_{2^u-1}) & \\
\hline
A_L & & & &
\end{array}
$$

where $[\,\omega'_0 \ \dots \ \omega'_{2^l-1}\,] := Val(A_L)$ and $[\,\omega''_0 \ \dots \ \omega''_{2^u-1}\,] := Val(A_U)$.

For example let $A = [\,hot\ rain\ snow\ wet\,]$ and τ be a world on A as given by the table below. If we put $A_L := [\,hot\ snow\,]$ we get $A_U = A \backslash A_L = [\,rain\ wet\,]$ and the double table of τ on A_L and A_U is well–defined:

hot	*rain*	*snow*	*wet*	
?	?	?	?	!
!	?	?	?	!
?	!	?	?	?
!	!	?	?	?
?	?	!	?	?
!	?	!	?	?
?	!	!	?	?
!	!	!	?	?
?	?	?	!	?
!	?	?	!	?
?	!	?	!	?
!	!	?	!	!
?	?	!	!	!
!	?	!	!	?
?	!	!	!	?
!	!	!	!	?

		?	!	?	!	*rain*
		?	?	!	!	*wet*
?	?	!	?	?	?	
!	?	!	?	?	!	
?	!	?	?	!	?	
!	!	?	?	?	?	
hot	*snow*					

So, given a theory or a table, there is a unique double table for each specified left (or upper) atom list. On the other hand, each double table represents a unique theory and table.

3 Operations on worlds and atoms lists

Tables and double tables are proper representations for worlds, but as the list of atoms increases, their size grows exponentially. Another way of representing worlds is by means of algebraic expressions (i.e. formulas): taking a subset of basic or *atomic* worlds, and defining a couple of operations, we can generate all possible worlds by recursively applying these operations on atomic worlds and results of previous operations.

So the most important constants and operations we are going to define are:
- the set of *atomic worlds*,
- the *zero* and *unit world*
- the *expansion* ($\|$) and two *reductions* ($\Uparrow$,$\Downarrow$) to in– and decrease the atom sets of worlds, and
- the *boolean junctions* ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$).

The theory of the last example can then be written in a more compact form as say

$$[\,\neg[\,rain \wedge snow\,] \wedge [\,wet \leftrightarrow [\,rain \vee snow\,]\,] \wedge [\,rain \rightarrow hot\,] \wedge [\,snow \rightarrow \not{hot}\,]\,]$$

what can be read as

> “It cannot rain and snow, it is wet if and only if it rains or snows, if it rains, it is hot, and if it snows, it is not hot.”

3.1 Atomic worlds

For every atom α there is a unique **atomic world** (of α), given by the table

α	
?	?
!	!

We don't want to introduce a new notation for this atom world, so we simply write it as α as well. Note that there is a bijection between the set $\mathcal{A}$ of atoms and the set of all the atomic worlds generated from $\mathcal{A}$ and that should make this convention save against confusions.

3.2 The zero and unit world

If A is the empty set or list $[\ ]$ of atoms, there is exactly one valuation of A, namely the valuation $\omega = [\,]$. So there are two well–defined worlds on A, namely

- the **zero world**, written $\perp$, which is given by

$$\perp : Val([\,]) \longrightarrow \mathcal{B} \quad \text{with} \quad \perp(\omega) := ?$$

  and its table

	?

- the **unit world**, written $\top$, which is given by

$$\top : Val([\,]) \longrightarrow \mathcal{B} \quad \text{with} \quad \top(\omega) := !$$

  and its table

	!

3.3 Expansion

Let τ be a world on an atom list A and let $B = [\,\alpha'_1 \dots \alpha'_n\,]$ a second atom list. The **expansion of τ with** B is written

$$[\,\tau \parallel B\,] \quad \text{or} \quad [\,\tau \parallel \alpha'_1\ \dots\ \alpha'_n\,]$$

and it is defined to be the theory on $A \cup B$ that is *equivalent* to the original theory τ. A proper definition of the equivalence relation is given below and it is introduced by means of the expansion operation. For now we only need to describe how to constuct the new world: let $B' := B \backslash A$, so B' and A are disjunct and $A \cup B' = A \cup B$, then

$$[\,\tau \parallel B\,] : (A \cup B') \longrightarrow \mathcal{B} \quad \text{with} \quad [\,\tau \parallel B\,](\omega \cdot \omega') := \tau(\omega)$$

for every $\omega \in Val(A)$ and $\omega' \in Val(B')$.

This definition can be demonstrated with the use of double tables. Given $A := [\,a\ c\,]$ and $B := [\,a\ b\ d\,]$ and a world τ on A defined by the table

a	c	
?	?	!
!	?	?
?	!	!
!	!	!

In order to construct $[\,\tau \parallel B\,]$ we first subtract A from B and get $B' := B\backslash A = [\,b\ d\,]$. Then we build the double table on A and B', in which each row has the same bit values than the original table, but multiplied over the whole row. Finally this double table can be transformed into its table form, and that's the result $[\,\tau \parallel B\,]$. For our example that means

a	c	
?	?	!
!	?	?
?	!	!
!	!	!

becomes

		?	!	?	!	b
		?	?	!	!	d
?	?	!	!	!	!	
!	?	?	?	?	?	
?	!	!	!	!	!	
!	!	!	!	!	!	
a	c					

which is

a	b	c	d	
?	?	?	?	!
!	?	?	?	?
?	!	?	?	!
!	!	?	?	?
?	?	!	?	!
!	?	!	?	!
?	!	!	?	!
!	!	!	?	!
?	?	?	!	!
!	?	?	!	?
?	!	?	!	!
!	!	?	!	?
?	?	!	!	!
!	?	!	!	!
?	!	!	!	!
!	!	!	!	!

3.4 Boolean junctions of worlds

For every world τ on A the **negation of** τ is the world $\neg\tau$ on A, defined by

$$\neg\tau : Val(A) \longrightarrow \mathcal{B} \quad \text{with} \quad \neg\tau_1(\omega) := \begin{cases} ? & \text{if } \tau(\omega) = \,! \\ ! & \text{if } \tau(\omega) = \,? \end{cases}$$

Let $\tau_1, \ldots, \tau_n$ be worlds, $n \geq 0$, all on the same atom list A. We define the following new worlds on A, as results of operations on the given ones:

$$[\,\tau_1 \wedge \cdots \wedge \tau_n\,] : Val(A) \longrightarrow \mathcal{B} \quad \text{with} \quad [\,\tau_1 \wedge \cdots \wedge \tau_n\,](\omega) := \begin{cases} ? & \text{if } ? \in \{\tau_1(\omega), \ldots, \tau_n(\omega)\} \\ ! & \text{if } ? \notin \{\tau_1(\omega), \ldots, \tau_n(\omega)\} \end{cases}$$

$$[\,\tau_1 \vee \cdots \vee \tau_n\,] : Val(A) \longrightarrow \mathcal{B} \quad \text{with} \quad [\,\tau_1 \vee \cdots \vee \tau_n\,](\omega) := \begin{cases} ? & \text{if } ! \notin \{\tau_1(\omega), \ldots, \tau_n(\omega)\} \\ ! & \text{if } ! \in \{\tau_1(\omega), \ldots, \tau_n(\omega)\} \end{cases}$$

$$[\,\tau_1 \rightarrow \tau_2\,] : Val(A) \longrightarrow \mathcal{B} \quad \text{with} \quad [\,\tau_1 \rightarrow \tau_2\,](\omega) := \begin{cases} ? & \text{if } \tau_1(\omega) \not< \tau_2(\omega) \\ ! & \text{if } \tau_1(\omega) < \tau_2(\omega) \end{cases}$$

$$[\,\tau_1 \leftrightarrow \tau_2\,] : Val(A) \longrightarrow \mathcal{B} \quad \text{with} \quad [\,\tau_1 \leftrightarrow \tau_2\,](\omega) := \begin{cases} ? & \text{if } \tau_1(\omega) \neq \tau_2(\omega) \\ ! & \text{if } \tau_1(\omega) = \tau_2(\omega) \end{cases}$$

More general, for every list of worlds $\tau_1, \ldots, \tau_n$ with $n \geq 0$ we define

$$\begin{aligned}
[\,\tau_1 \wedge \cdots \wedge \tau_n\,] &:= [\,[\,\tau_1 \parallel A\,] \wedge \cdots \wedge [\,\tau_n \parallel A\,]\,] && \text{with} \quad A := \bigcup_{i=1}^{n} @\tau_i \\
[\,\tau_1 \vee \cdots \vee \tau_n\,] &:= [\,[\,\tau_1 \parallel A\,] \vee \cdots \vee [\,\tau_n \parallel A\,]\,] && \text{with} \quad A := \bigcup_{i=1}^{n} @\tau_i \\
[\,\tau_1 \rightarrow \tau_2\,] &:= [\,[\,\tau_1 \parallel A\,] \rightarrow [\,\tau_2 \parallel A\,]\,] && \text{with} \quad A := @\tau_1 \cup @\tau_2 \\
[\,\tau_1 \leftrightarrow \tau_2\,] &:= [\,[\,\tau_1 \parallel A\,] \leftrightarrow [\,\tau_2 \parallel A\,]\,] && \text{with} \quad A := @\tau_1 \cup @\tau_2
\end{aligned}$$

For example, let $\tau_1 := [\,a \parallel c\,]$ and $\tau_2 := [\,c \parallel b\,]$. Their tables are

a	c	τ_1
?	?	?
!	?	!
?	!	?
!	!	!

and

b	c	τ_2
?	?	?
!	?	?
?	!	!
!	!	!

Their negations are

a	c	$\neg\tau_1$
?	?	!
!	?	?
?	!	!
!	!	?

and

b	c	$\neg\tau_2$
?	?	!
!	?	!
?	!	?
!	!	?

We put $A := @\tau_1 \cup @\tau_2 = [\, a\; c\,] \cup [\, b\; c\,] = [\, a\; b\; c\,]$ and generate $[\,\tau_1 \parallel A\,]$ and $[\,\tau_2 \parallel A\,]$:

a	b	c	$[\,\tau_1 \parallel A\,]$
?	?	?	?
!	?	?	!
?	!	?	?
!	!	?	!
?	?	!	?
!	?	!	!
?	!	!	?
!	!	!	!

and

a	b	c	$[\,\tau_2 \parallel A\,]$
?	?	?	?
!	?	?	?
?	!	?	?
!	!	?	?
?	?	!	!
!	?	!	!
?	!	!	!
!	!	!	!

We then obtain the worlds $[\,\tau_1 \wedge \tau_2\,]$, $[\,\tau_1 \vee \tau_2\,]$, $[\,\tau_1 \rightarrow \tau_2\,]$, $[\,\tau_1 \leftrightarrow \tau_2\,]$:

a	b	c	$[\,\tau_1 \wedge \tau_2\,]$
?	?	?	?
!	?	?	?
?	!	?	?
!	!	?	?
?	?	!	?
!	?	!	!
?	!	!	?
!	!	!	!

a	b	c	$[\,\tau_1 \vee \tau_2\,]$
?	?	?	?
!	?	?	!
?	!	?	?
!	!	?	!
?	?	!	!
!	?	!	!
?	!	!	!
!	!	!	!

a	b	c	$[\,\tau_1 \rightarrow \tau_2\,]$
?	?	?	!
!	?	?	?
?	!	?	!
!	!	?	?
?	?	!	!
!	?	!	!
?	!	!	!
!	!	!	!

a	b	c	$[\,\tau_1 \leftrightarrow \tau_2\,]$
?	?	?	!
!	?	?	?
?	!	?	!
!	!	?	?
?	?	!	?
!	?	!	!
?	!	!	?
!	!	!	!

Results of operations can be used for new operations, so in this way we can recursively generate the worlds and tables of more complex expressions like the earlier mentioned

$$[\,\neg[\,rain \wedge snow\,] \wedge [\,wet \leftrightarrow [\,rain \vee snow\,]\,] \wedge [\,rain \rightarrow hot\,] \wedge [\,snow \rightarrow \not hot\,]\,]$$

3.5 Infimum and supremum reduction

Let τ be a world on A and B a second atom list.

The *expansion* $[\,\tau \parallel B\,]$ of τ with B was defined to be the world, uniquely defined by two properties: it must be equivalent to τ and its atom list should be $A \cup B$.

Accordingly, we could define a *reduction* of τ with B by saying that the result must be equivalent to τ and have the atom list $A \cap B$. But there is a problem: such a world doesn't exist in general.

Comparing this phenomenon with integer arithmetic, the *multiplication* $i \cdot j$ is always well–defined, but the *division* i/j isn't, as for example in $7/4$, where the result 1.75 is not an integer anymore. But we could take the closest integers, here the lower 1 and upper 2 as results of two modified operations, called *infimum division* $\nearrow$ and *supremum division* $\swarrow$, e.g. $1 = 7 \nearrow 4 \leq 7/4 \leq 7 \swarrow 4 = 2$ and $9 = 10 \nearrow 3 \leq 10/3 \leq 10 \swarrow 3 = 10$.

Back to worlds (where the role of the integer order $\leq$ is taken by the order relation $\Rightarrow$), we also define two reductions, the closest lower and upper reduction, called **infimum reduction** $[\,\tau \Uparrow B\,]$ **of** τ **onto** B and **supremum reduction** $[\,\tau \Downarrow B\,]$ **of** τ **onto** B. It turns out to be more convenient for later purposes, that the new atom list is not set to $A \cap B$, but to B itself (although the results of both approaches are always equivalent). So an atomic "reduction" not necessarily "reduces" the atom list, but may also introduce new atoms.

The introduction of these operations can again be well demonstrated by using double tables: First, put $A' := A \backslash B$ and construct the double table of $[\,\tau \parallel B\,]$ with left atoms B and upper atoms A'. For the infimum reduction, the bit values of each row have to be replaced by their conjunction. For the supremum reduction, the bit values of each row have to be replaced by their disjunction. For example, let us reduce the last example theory τ onto $B := [\,\mathit{hot}\ \mathit{rain}\,]$. The table on B and $A' = [\,\mathit{snow}\ \mathit{wet}\,]$ is given by

		?	!	?	!	*snow*
		?	?	!	!	*wet*
?	?	!	?	?	!	
!	?	!	?	?	?	
?	!	?	?	?	?	
!	!	?	?	!	?	
hot	*rain*					

The infimum reduction $[\,\tau \Uparrow B\,]$ is then given by

hot	*rain*	$[\,\tau \Uparrow B\,]$
?	?	$[\,! \wedge ? \wedge ? \wedge !\,]$
!	?	$[\,! \wedge ? \wedge ? \wedge ?\,]$
?	!	$[\,? \wedge ? \wedge ? \wedge ?\,]$
!	!	$[\,? \wedge ? \wedge ! \wedge ?\,]$

which is

hot	*rain*	$[\,\tau \Uparrow B\,]$
?	?	?
!	?	?
?	!	?
!	!	?

and the infimum reduction $[\,\tau \Downarrow B\,]$ is then given by

hot	*rain*	$[\,\tau \Downarrow B\,]$
?	?	$[\,! \vee ? \vee ? \vee !\,]$
!	?	$[\,! \vee ? \vee ? \vee ?\,]$
?	!	$[\,? \vee ? \vee ? \vee ?\,]$
!	!	$[\,? \vee ? \vee ! \vee ?\,]$

which is

hot	*rain*	$[\,\tau \Downarrow B\,]$
?	?	!
!	?	!
?	!	?
!	!	!

A formal definition for arbitrary worlds τ and atom lists B are given by:
$[\,\tau \Uparrow B\,] : \mathit{Val}(B) \longrightarrow \mathcal{B}$ with

$$[\,\tau \Uparrow B\,](\omega) := [\,\tau(\omega \cdot \omega_1') \wedge \cdots \wedge \tau(\omega \cdot \omega_n')\,] \quad \text{for} \quad [\,\omega_1' \ \ldots \ \omega_n'\,] := \mathit{Val}(@\tau \backslash B)$$

$[\,\tau \Downarrow B\,] : \mathit{Val}(B) \longrightarrow \mathcal{B}$ with

$$[\,\tau \Downarrow B\,](\omega) := [\,\tau(\omega \cdot \omega_1') \vee \cdots \vee \tau(\omega \cdot \omega_n')\,] \quad \text{for} \quad [\,\omega_1' \ \ldots \ \omega_n'\,] := \mathit{Val}(@\tau \backslash B)$$

3.6 Remark about the chosen notations

The titles of these two new operations come from the fact that, of all theories on the new atom list B, $[\,\tau \Uparrow B\,]$ is the *greatest lower bound* (i.e. the infimum) and $[\,\tau \Downarrow B\,]$ is the *least upper bound* (i.e. the supremum), with respect to the *subvalence relation* $\Rightarrow$ defined below. So, $[\,\tau \Uparrow B\,] \Rightarrow \tau \Rightarrow [\,\tau \Downarrow B\,]$ always holds.

In lattice theory *infimum* is synonymous to *conjunction*, and that stipulates the notation $\Uparrow$ as "$\wedge$ and $\|$". The same holds for *supremum*, *disjunction* and the $\Downarrow$ symbol.

The $\|$ symbol itself is chosen due to its similarities with the symbolism $f|B$, which usually denotes the reduction of the domain A of a function $f : A \longrightarrow C$ onto a subset $B \subseteq A$.

4 Relations between worlds

On the set of all worlds (on $\mathcal{A}$) we define the following relations.

- The **atomic order relations**:
  For every pair τ_1, τ_2 of worlds we put

$$\tau_1 \mathrel{\hat{\subseteq}} \tau_2 \quad :\text{iff} \quad @\tau_1 \subseteq @\tau_2$$

  saying that τ_1 is **subatomic** to τ_2, and

$$\tau_1 \mathrel{\hat{=}} \tau_2 \quad :\text{iff} \quad @\tau_1 = @\tau_2$$

  saying that τ_1 and τ_2 are **equiatomic**.

- The **boolean order relations**:
  For every pair τ_1, τ_2 of worlds we put

$$\begin{array}{lll} \tau_1 \Rightarrow \tau_2 & \text{:iff} & [\,\tau_1 \parallel A\,] < [\,\tau_2 \parallel A\,], \text{ for all } \omega \in Val(A) \text{ with } A := @\tau_1 \cup @\tau_2 \\ & \text{:iff} & [\,\tau_1 \rightarrow \tau_2\,] \text{ is valid} \end{array}$$

  saying that τ_1 is **subvalent** to τ_2, and

$$\begin{array}{lll} \tau_1 \Leftrightarrow \tau_2 & \text{:iff} & [\,\tau_1 \parallel A\,] = [\,\tau_2 \parallel A\,], \text{ for all } \omega \in Val(A) \text{ with } A := @\tau_1 \cup @\tau_2 \\ & \text{:iff} & [\,\tau_1 \leftrightarrow \tau_2\,] \text{ is valid} \end{array}$$

  saying that τ_1 and τ_2 are **equivalent**.
- The **theory order relations**:
  For every pair τ_1, τ_2 of worlds we put

$$\tau_1 \mathrel{\hat{\Rightarrow}} \tau_2 \quad \text{:iff} \quad \tau_1 \mathrel{\hat{\subseteq}} \tau_2 \text{ and } \tau_1 \Rightarrow \tau_2$$

  saying that τ_1 is **bisubvalent** to τ_2, and

$$\tau_1 \mathrel{\hat{\Leftrightarrow}} \tau_2 \quad \text{:iff} \quad \tau_1 \mathrel{\hat{=}} \tau_2 \text{ and } \tau_1 \Leftrightarrow \tau_2$$

  saying that τ_1 and τ_2 are **biequivalent**.

Each of the three relations $\hat{\subseteq}, \Rightarrow, \hat{\Rightarrow}$ is a *quasi–order* relation, i.e. it is reflexive and transitive on the set of all worlds. Each of the three relations $\hat{=}, \Leftrightarrow, \hat{\Leftrightarrow}$ is the according *equivalence relation* of the given quasi–order, i.e. for example $\tau_1 \mathrel{\hat{=}} \tau_2$ iff $\tau_1 \mathrel{\hat{\subseteq}} \tau_2$ and $\tau_2 \mathrel{\hat{\subseteq}} \tau_1$.

Note, that for the algebra of worlds the biequivalence is the identity relation, i.e.

$$\tau_1 \mathrel{\hat{\Leftrightarrow}} \tau_2 \quad \text{iff} \quad \tau_1 = \tau_2$$

Relations can be defined as binary operations with $\mathcal{B}$ as codomain. For example, $\Rightarrow$ can be defined as an operation $[\cdots \Rightarrow \ldots]$ that takes two worlds τ_1 and τ_2 and returns a bit value

$$[\,\tau_1 \Rightarrow \tau_2\,] := \begin{cases} ! & \text{if } \tau_1 \Rightarrow \tau_2 \text{, i.e. if } \tau_1 \text{ if subvalent to } \tau_2 \\ ? & \text{if } \tau_1 \not\Rightarrow \tau_2 \text{, i.e. if } \tau_1 \text{ is not subvalent to } \tau_2 \end{cases}$$

5 More operations

5.1 Negative and positive atoms

An atom α is *negative* or *redundant* in a world τ if it doesn't matter what bit value it takes: for every $\omega \in Val(@\tau\backslash[\,\alpha\,])$, $\omega \cdot [\,\alpha/?\,]$ is a possible state iff $\omega \cdot [\,\alpha/!\,]$ is a possible state, too. The term "redundant" emphasizes the fact that, if α is redundant in τ, there is an equivalent world without α.

Using double tables, we can determine if α is negative in τ by representing it with the left atoms $@\tau\backslash[\,\alpha\,]$ and upper atoms $[\,\alpha\,]$. α is indeed negative in τ iff the two given bit values are the same in each row.

For example, let $\tau := [\,a \vee [\,c \wedge \neg c\,] \vee b\,]$. Obviously, c is redundant / negative in τ, because $\tau \Leftrightarrow [\,a \vee b\,]$. The double table with c as the only upper atom shows that as well, each row is made of just one bit value:

		?	!	c
?	?	?	?	
!	?	!	!	
?	!	!	!	
!	!	!	!	
a	b			

For a full definition, let us put for every world τ and every atom α (where not necessarily $\alpha \in @\tau$ has to hold):
- α is **negative** or **redundant for** τ iff

$$[\,\tau \Uparrow [\,@\tau\backslash[\,\alpha\,]\,]\,] \Leftrightarrow [\,\tau \Downarrow [\,@\tau\backslash[\,\alpha\,]\,]\,]$$

- α is **positive** or **irredundant for** τ iff

$$[\,\tau \Uparrow [\,@\tau \backslash [\,\alpha\,]\,]\,] \not\Leftrightarrow [\,\tau \Downarrow [\,@\tau \backslash [\,\alpha\,]\,]\,]$$

If α is an atom in τ, we say that α is negative / positive **in** τ.

And for every world τ we introduce the notation

$$\begin{array}{rcl} -@\tau & := & \text{the ordered list of all negative atoms in } \tau \\ +@\tau & := & \text{the ordered list of all positive atoms in } \tau \end{array}$$

Obviously, for every world τ

$$\begin{array}{rcl} -@\tau \cup +@\tau & = & @\tau \\ -@\tau \cap +@\tau & = & [\,] \end{array}$$

For example, for $\tau := [\,a \vee [\,c \wedge \neg c\,] \vee b\,]$ there is $-@\tau = [\,c\,]$ and $+@\tau = [\,a\,b\,]$.

5.2 Standard reduction

Due to the just given definition, every world τ has an equivalent world which is only made of the positive atoms of τ. Such a world is called a **standard reduction** of τ and written $@|\tau$. We put

$$@|\tau := [\,\tau \Uparrow +@\tau\,] = [\,\tau \Downarrow +@\tau\,]$$

Consequently, for every world τ there is

$$\tau = [\,@|\tau \parallel -@\tau\,]$$

This theorem is the basis for the so–called *extended normal forms* of the Bucanon program. If τ is a theory formula, then the PDNF τ_d and PCNF τ_c of τ are made of irredundant atoms only, so both forms are standard reductions of τ. In order to get not only an equivalent, but also an equiatomic normal form of τ, the negative atoms $-@\tau$ are attached again, so $[\,\tau_d \parallel -@\tau\,]$ is the XPDNF (extended prime disjunctive normal form) of τ and $[\,\tau_c \parallel -@\tau\,]$ is its XPDNF (extended prime conjunctive normal form).

For example, for $\tau := [\,a \vee [\,c \wedge \neg c\,] \vee b\,]$

$$\begin{array}{rl} [\,[\,\wedge a\,] \vee [\,\wedge b\,]\,] & \text{is the PDNF of } \tau\text{, so} \\ [\,[\,[\,\wedge a\,] \vee [\,\wedge b\,]\,] \parallel c\,] & \text{is the XPDNF of } \tau \\ [\,\wedge [\,a \vee b\,]\,] & \text{is the PCNF of } \tau\text{, so} \\ [\,[\,\wedge [\,a \vee b\,]\,] \parallel c\,] & \text{is the XPCNF of } \tau \end{array}$$